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Effects of the ρ Meson on Nucleon Form Factors*

J. S. BALL AND D. Y. WONG
University of California at San Diego, La Jolla, California
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The isovector charge and anomalous magnetic moment form factors of the nucleon are calculated assuming that the low-energy part of the spectral function is determined by the two-pion intermediate state and the high-energy part of the spectral function can be represented by a subtraction constant. The results are compared with experimental analysis in the form of a pole plus a subtraction constant for each form factor. The spectral function is given in terms of the pion form factor ($2\pi\rightarrow\gamma$) and the $N\bar{N}\rightarrow 2\pi$ amplitudes. The phase of $2\pi\rightarrow\gamma$ as well as $N\bar{N}\rightarrow 2\pi$ are determined in terms of $\pi\pi$ p-wave phase shift which is adjusted to fit the observed ρ-resonance with a mass of 760 MeV and a full width of $\sim$130 MeV. In the calculation of the $N\bar{N}\rightarrow 2\pi$ amplitudes, the exchange of a nucleon and a (3-3) resonance are included as Regge poles. Two parameters are introduced in the Regge pole description. These parameters are adjusted to fit two constants a_1 and a_2 obtained from the experimental analysis of the form factors. The effective mass of the two-pion state which results from the present calculation is approximately 600 to 650 MeV (well below the ρ mass), in good agreement with the experimental determination.

I. INTRODUCTION

EXPERIMENTS determining the isovector nucleon electromagnetic form factors have stimulated much theoretical work.[1] In particular, the large value of the electromagnetic radius determined in these experiments led Frazer and Fulco[2] to conjecture that the exchange of a pion-pion p-wave resonance was the most important mechanism in producing the nucleon structure. They predicted that the pion-pion resonance would have t, the total energy squared, in the range from 10 to 16. (We employ units in which $\hbar=c=\mu_\pi=1$ throughout this paper.) Subsequent work by Bowcock, Cottingham, and Lurié shifted this value to the neighborhood of 20.[3] Both of these calculations had some ambiguities, particularly in their treatment of the amplitude $N\bar{N}\rightarrow\pi\pi$ which is needed to obtain the imaginary part of the form factors.

The subsequent discovery of a pion-pion resonance at $t=29$ seems to be a good confirmation of the theoretical prediction,[4] but it leaves one important question unanswered; namely, does this resonance with a higher mass produce a nucleon form factor consistent with experiment as was originally conjectured by FF? It is this question which this paper discusses. The fact that the pion-pion scattering amplitude is fairly well-known in the neighborhood of the resonance gives us an important advantage over FF; namely, that no arbitrary parameters need be introduced to describe the pion form factor in this region. In the present work, we introduce two parameters to account for the uncertainty in the $N\bar{N}\rightarrow\pi\pi$ amplitudes. It is shown that these two parameters can be adjusted to fit the experimental isovector form factors.

The experimental situation at this time is that the data can be represented by the following formulas:

$$G_1{}^v=\left(1-a_1+\frac{a_1}{1-t/t_1}\right)\left(\frac{e}{2}\right)$$

$$G_2{}^v=1.83\left(1-a_2+\frac{a_2}{1-t/t_2}\right)\left(\frac{e}{2m}\right) \qquad (1.1)$$

where $G_1{}^v$ is the charge form factor and $G_2{}^v$ is the anomalous magnetic moment form factor. De Vries, Hofstadter, and Herman have given the following values to the parameters: $a_1\cong 0.92$, $a_2\cong 1.15$, $t_1=t_2\cong 18$.[5] The pole term in Eq. (1.1) is attributed as the contribution from the pion-pion resonance while the constant term can be considered to be the contributions from higher mass intermediate states. It is, then, the pole terms in Eq. (1.1) with which the calculated pion-pion contribution to the form factors should agree.

Our procedure is as follows: In the next section we formulate the calculation of the $N\bar{N}\rightarrow\pi\pi$ amplitude in terms of the π-π scattering amplitude and the most important states in low-energy pion-nucleon scattering; namely, the single nucleon pole and the "3-3" resonance. A cutoff procedure is introduced which is consistent with the "Regge" behavior of the particle and resonance states. The "Regge" behavior for each of these states is controlled by a single parameter. In the following section we discuss several different approximations to the pion-pion scattering amplitude all of which fit the experimentally observed resonance width and energy.

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* Work supported by the U. S. Atomic Energy Commission.

[1] See C. de Vries, R. Hofstadter, and R. Herman, Phys. Rev. Letters 8, 381 (1962), for a complete list of references.

[2] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959); and Phys. Rev. 117, 1609 (1960), hereafter referred to as FF.

[3] J. Bowcock, W. Cottingham, and D. Lurié, Phys. Rev. Letters 5, 386 (1960).

[4] A. Erwin, R. March, W. Walker, and E. West, Phys. Rev. Letters 6, 628 (1961); J. A. Anderson, V. X. Bang, P. G. Burke, D. D. Carmony, and N. Schmitz, *ibid.* 6, 365 (1961); D. Stonehill, C. Baltay, H. Courant, W. Fickinger, E. C. Fowler, H. Kraybill, J. Sandweiss, J. Sanford, and H. Taft, *ibid.* 6, 624 (1960); A. R. Erwin, R. March, W. D. Walker, and E. West, *ibid.* 6, 628 (1960); J. G. Rushbrooke and D. Radojičić, *ibid.* 5, 567 (1960); E. Pickup, F. Ayer, and E. O. Salant, *ibid.* 5, 161 (1960); C. Alff, D. Berley, D. Colley, N. Gelfand, U. Nauenberg, D. Miller, J. Schultz, J. Steinberger, and T. H. Tan, *ibid.* 9, 322 (1962).

[5] C. de Vries, R. Hofstadter, and R. Herman, Phys. Rev Letters 8, 381 (1962).

In Sec. IV, the form factors are calculated and the parameters of the cutoff are adjusted to fit a_1 and a_2. The resulting form factors are then shown to have an effective mass (t_1 and t_2) substantially smaller than the mass of the ρ-resonance, consistent with the fit to experimental data of the type given in Eq. (1.1). These results are insensitive to the details of different π-π amplitudes employed.

II. THE $\bar{N}\bar{N}\to\pi\pi$ AMPLITUDE

The relation between the $N\bar{N}\to\pi\pi$ amplitude and the nucleon electromagnetic structure can be seen from the following spectral representations which give the two-pion contribution to the isovector nucleon form factors:

$$G_1{}^\rho=\frac{1}{\pi}\int_4^\infty\frac{dt'\,g_1{}^\rho(t')}{t'-t},\quad G_2{}^\rho=\frac{1}{\pi}\int_4^\infty\frac{dt'\,g_2{}^\rho(t')}{t'-t}.\qquad(2.1)$$

The spectral functions $g_i{}^\rho$ are given by[2]

$$g_i{}^\rho=-eq^3t^{-1/2}F_\pi{}^*(t)\Gamma_i(t);\quad i=1,2,\qquad(2.2)$$

where the Γ_i's are the two $J+1$ odd-parity scattering amplitudes for $N\bar{N}\to\pi\pi$ defined by FF, F_π is the pion form factor, and $q=(t/4-1)^{1/2}$ is the pion momentum. The fact that F_π is peaked in the neighborhood of the pion-pion resonance means that any attempt to calculate the nucleon form factors must include an approximation for the $N\bar{N}\to\pi\pi$ amplitudes which is reliable in the region of the pion-pion resonance.

The unitarity condition on the $N\bar{N}\to\pi\pi$ amplitudes requires Γ_i to have the phase of π-π scattering in the region $4<t<16$; we will assume that the effects of inelastic scattering are small so that this phase condition will continue to be approximately valid at higher energies. In addition to the right-hand singularities given by the unitary conditions, the functions Γ_i will have left-hand cuts which are related to the physical singularities of pion-nucleon scattering. If we define $\Gamma_i{}^B$ to be the result obtained by carrying out the integration over the left-hand cuts of Γ_i, we can write the following solutions for the Γ's which will satisfy the appropriate phase condition for $t>4$:

$$\Gamma_i(t)=\Gamma_i{}^B(t)+\frac{1}{\pi D(t)}\int_4^\infty\frac{dt'\,N(t')\Gamma_i{}^B(t')}{t'-t}\left(\frac{t'-4}{t'}\right)^{1/2},\qquad(2.3)$$

where N/D is the following function of the π-π p-wave phase shift δ_π:

$$\frac{N}{D}=\left(\frac{t}{t-4}\right)^{1/2}\sin\delta_\pi\,e^{i\delta_\pi}.\qquad(2.4)$$

The function $1/D$ has the phase of pion-pion scattering while N is regular for $t>4$. For $t<4$, $1/D$ is regular. Using these facts and expressing the integral in Eq. (2.3) as a principal value integral plus an imaginary part it is easily seen that the Γ's have the phase δ_π for $t>4$. The left-hand singularities are obviously correct as D is regular for $t<4$ and the singularities of the Γ^B's are identical to those of the Γ's by definition.

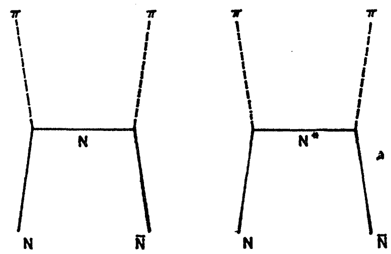

FIG. 1. Diagrams representing the exchange of a nucleon and an excited nucleon (3-3 resonance).

In principle, the discontinuities across the left-hand singularities can be calculated from the pion-nucleon scattering amplitudes by using the crossing relations. In practice, the difficulty in the calculation of these discontinuities is that, while the energy variable s is in the physical region for $\pi-N$ scattering, t is outside of its physical region; hence, experimental information on $\pi-N$ scattering cannot be used directly. However, as long as the polynomial expansion for the $\pi-N$ scattering amplitudes converges, this expansion may be used to continue the $\pi-N$ amplitudes to the desired values of t. FF have shown that this procedure allows the calculation of the discontinuities for $t>-26$ provided the complete partial wave expansion were known. The fact that low-energy pion-nucleon scattering is dominated by the 3-3 resonance means that the nearby portion of the left-hand cut has discontinuities approximately given in terms of the 3-3 amplitude in addition to the nucleon pole contribution. It is the uncertainty about the rest of the left-hand singularities that makes the calculation of Γ^B ambiguous in the region $t>4$. However, we still expect the nucleon pole and the 3-3 amplitude to be the dominant terms in the low-t region. For large values of t, the nucleon and the 3-3 resonance contributions may still dominate the Γ^B amplitude provided they are treated as Regge poles rather than the usual poles in fixed angular momentum amplitudes for πN scattering.[6] Due to the lack of detailed knowledge of the behavior of Regge poles in πN scattering, we introduce two parameters in modifying the nucleon pole and the 3-3 resonance term to fit the asymptotic behavior given by the Regge pole hypothesis. First, we calculate the two pole terms shown in Fig. 1, treating the N^* as a single particle with mass m_{33} and spin 3/2 (If the Γ's are calculated from these terms, the same result is obtained as would be obtained by first calculating the left-hand discontinuity including only the nucleon pole and the 3-3 state but assuming that the polynomial expansion is valid for all t and then integrating over the left-hand cut). Then we multiply the nucleon term which is a pole at $s=m^2$ by the factor $\exp[C_N(s-m^2)\ln(t/4)]$ and the 3-3 term, a pole at

[6] For a discussion of Regge poles see, for example, G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 394 (1961).

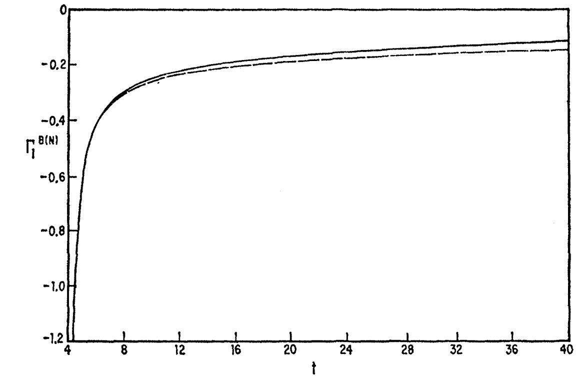

FIG. 2. The effect of the "Regge" cutoff for the nucleon pole term $\Gamma_1^{B(N)}$. The solid curve is $\Gamma_1^{B(N)}$ for $C_N=0.001$, while the dashed curve is for $C_N=0$.

$s=m_{33}^2$, by $\exp[C_{33}(s-m_{33}^2)\ln(t/4)]$. Here, s is the energy variable for $\pi-N$ scattering. The two parameters C_N and C_{33} are positive as implied by the convergent behavior of Regge poles. These exponential factors become unity when evaluated at the appropriate pole in the πN channel as well as at $t=4$ where the ordinary pole terms are expected to give a good approximation for Γ^B. Having obtained the modified pole terms, we carry out the partial-wave projection in the channel $N\bar{N}\to\pi\pi$. The resulting expressions for the Γ^B's are as follows:

$$\Gamma_1^B(t)=\frac{g_r^2}{12\pi p^2}\int_{-1}^{+1}\frac{dz}{(m^2-s)}\times[(p^2+3m^2)P_2(z)-p^2]\left(\frac{t}{4}\right)^{C_N(s-m^2)}+\frac{2\gamma_{33}}{3p^2q}\int_{-1}^{+1}\frac{dz}{(m_{33}^2-s)}\{mpH(t)P_1(z)-\tfrac{1}{3}q\times[(p^2+3m^2)P_2(z)-p^2]F(t)\}\left(\frac{t}{4}\right)^{C_{33}(s-m_{33}^2)} \tag{2.5}$$

$$\Gamma_2^B(t)=-\frac{g_r^2m}{8\pi p^2}\int_{-1}^{+1}\frac{dz}{(m^2-s)}P_2(z)\left(\frac{t}{4}\right)^{C_N(s-m^2)}+\frac{\gamma_{33}}{3pq}\int_{-1}^{+1}\frac{dz}{(m_{33}^2-s)}\left[-H(t)P_1(z)+\frac{mq}{p}F(t)P_2(z)\right]\left(\frac{t}{4}\right)^{C_{33}(s-m_{33}^2)}, \tag{2.6}$$

where

$$H(t)=(m_{33}-m)(E_{33}+m)^2+\tfrac{3}{2}(m_{33}+m)(2k_{33}^2+t),$$
$$F(t)=-(E_{33}+m)^2+\tfrac{3}{2}(2k_{33}^2+t),$$

and s is the following function of z

$$s=m^2+1-\frac{t}{2}+2pqz.$$

The quantities p and q are the center-of-mass momentum for the nucleon and pion, respectively, in the $N\bar{N}\to\pi\pi$ channel. The nucleon energy and center-of-mass momentum at the 3-3 resonance are denoted by E_{33} and k_{33}. Here, g_r is the rationalized pion-nucleon coupling constant

$$g_r^2/4\pi=14,$$

and γ_{33} is the effective coupling constant for the 3-3 resonance, with a value of $\gamma_{33}=0.06$ which is obtained from the width of the 3-3 resonance assuming the width to be narrow.

It should be pointed out that the "Regge"-type modification we have introduced has no effect on the left-hand cut for $t>0$. For $t<0$, the discontinuity begins to deviate slowly from that given by crossing and has an oscillatory nature. For the present calculation, we need not concern ourselves with the left-hand region since we can evaluate Γ^B explicitly for $t\geq 4$.

In Fig. 2 we show the effect of this cutoff procedure by comparing $\Gamma_1^{B(N)}$, the contribution to Γ_1^B from the nucleon pole, with $C_N=0$ and $C_N=0.001$, a typical value obtained by fitting the form factors. The cutoff has very little effect on $\Gamma_2^{B(N)}$ until t becomes much larger than the resonance value. In Figs. 3 and 4, $\Gamma_{1,2}^{B(33)}$, the N^* contribution to $\Gamma_{1,2}^B$, is shown for $C_{33}=0$ and $C_{33}=0.01$, a typical value for this parameter.

III. THE PION-PION SCATTERING AMPLITUDE

In our treatment of the pion-pion scattering amplitude, we assume that the resonance is an elastic p-wave resonance. The possibility that small inelastic effects are present in this angular momentum state will not alter our results substantially. Since two parameters are determined experimentally (the position and width of the resonance),[4] we use a two-parameter phenomenological π-π solution, which is consistent with the unitary and analyticity requirements. This solution is obtained by solving the p-wave pion-pion N/D equa-

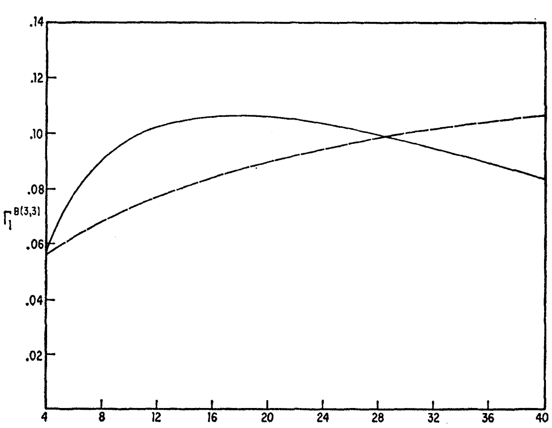

FIG. 3. The effect of the "Regge cutoff" for the 3-3 resonance term $\Gamma_1^{B(33)}$. The solid curve is $\Gamma_1^{B(33)}$ for $C_{33}=0.01$, while the dashed curve is for $C_{33}=0$.

TABLE I. Values of parameters for three acceptable fits to experimental form factors.

Parameters	Solutions I	II	III
A_1	0.051	0.072	0.204
A_2	0	0.032	0.175
ν_1	50000	5000	200
ν_2	$\cdots$	4.4	25
C_N	0.00085	0.0012	0.0018
C_{33}	0.008	0.011	0.02

tions assuming that the N function can be represented by a single pole. The position and residue of this pole are, then, adjusted to fit the observed position and width of the resonance. The resulting expressions for the pion-pion N and D functions are as follows:

$$N=\frac{\nu_1 A\nu}{\nu+\nu_1},$$
$$D=1-\nu\nu_1 AK(\nu,\nu_1),$$
$$K(\nu,\nu_1)=-\frac{2}{\pi}\frac{1}{\nu+\nu_1}[L(\nu_1)-L(-\nu)];\quad \nu=\tfrac{1}{4}t-1,$$
$$L(z)=\left(\frac{z}{z-1}\right)^{1/2}\ln[z^{1/2}+(z-1)^{1/2}], \tag{3.1}$$

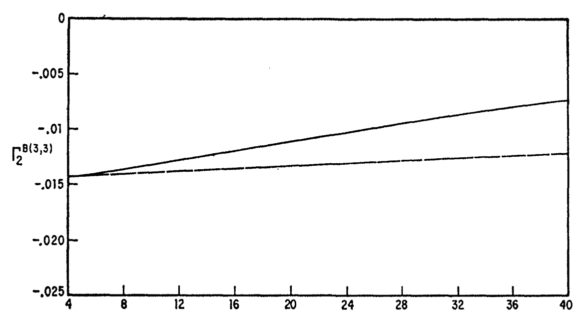

FIG. 4. The effect of the "Regge cutoff" for the 3-3 resonance term $\Gamma_2^{B(33)}$. The solid curve is $\Gamma_2^{B(33)}$ for $C_{33}=0.01$, while the dashed curve is for $C_{33}=0$.

and the relation of N and D to the pion-pion scattering amplitude is that given in Eq. (2.4). We find $\nu_1=5\times10^4$ and $A=0.051$ for a ρ resonance at 760 MeV with a half-width of 65 MeV.[7]

To investigate the dependence of the form factors on details of the pion-pion scattering amplitude other than width and position of the resonance, we also use a "two-pole" π-π solution. This solution is obtained by using a two-pole formula for the N function. The resulting solution depends on four parameters, two of which are adjusted to fit experiment; a third parameter is

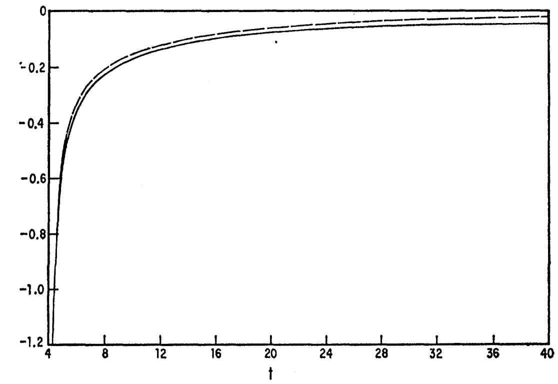

FIG. 5. The effect of unitarity in the $N\bar{N}\rightarrow\pi\pi$ reaction. The dashed curve is Γ_1^B which produces $D\Gamma_1$ represented by the solid curve.

eliminated by using a symmetry requirement on the pion-pion scattering amplitude. This method is based on the fact that the p-wave scattering amplitude for small negative t should be given by the fixed-t dispersion relation for pion-pion scattering in the channel in which s is the energy variable. The dependence of the form factors on the remaining parameter was investigated. The form of the two-pole solutions is as follows:

$$N=\left(\frac{\nu\nu_1A_1}{\nu+\nu_1}\right)-\left(\frac{\nu\nu_2A_2}{\nu+\nu_2}\right),$$
$$D=1-\nu[\nu_1A_1K(\nu,\nu_1)-\nu_2A_2K(\nu,\nu_2)]. \tag{3.2}$$

where $-\nu_1$, $-\nu_2$ are the positions of the two poles and A_1 and A_2 determine the residues. We considered two solutions of this type: one with poles at 5000 and 4.4 with $A_1=0.072$ and $A_2=0.032$; the other had poles at 200 and 25, with $A_1=0.204$ and $A_2=0.175$.

Both of these two solutions are very similar to the one pole solution in the resonance region with the important difference being the behavior for large positive t. It can be seen from Eq. (3.1) and (3.2) that N behaves as a linearly increasing function until t is of the order of magnitude of $4\nu_1$. The D function also behaves linearly for $t<4\nu_1$. The fact that these solutions have widely differing values of ν_1 means that the high-t contributions

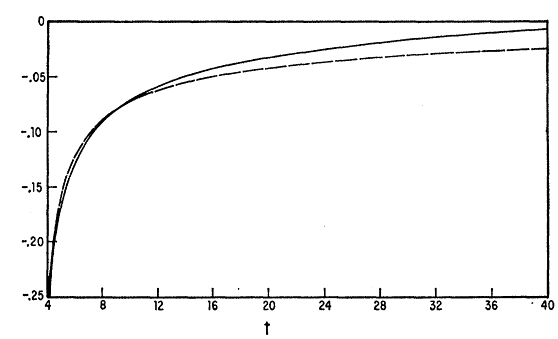

FIG. 6. The effect of unitarity of Γ_2. The dashed curve is Γ_2^B which produces $D\Gamma_2$ represented by the solid curve.

[7] We fit the shape of the resonance on the low-energy side of the maximum. The reason for this procedure is that our resonance formula is expected to be less reliable on the high-energy side of the peak.

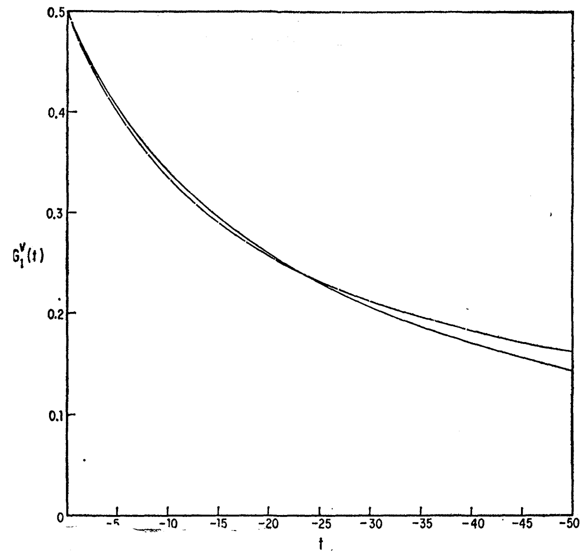

FIG. 7. The charge form factor. The dashed curve is our result for solution I. The solid curve is the fit to experiment (reference 5) given by Eq. (1.1) with $a_1=0.92$, $t_1=18$.

from the integrals in Eqs. (2.1) and (2.3) are substantially different in each case. However, as we will see below, the result for the form factor calculation has little dependence on which π-π solution we use. In Figs. 5 and 6 we show the effect of t-channel unitarity by comparing Γ_i^B with $D\Gamma_i$; $i=1$, 2, which are obtained from Eqs. (2.3) and (2.5).

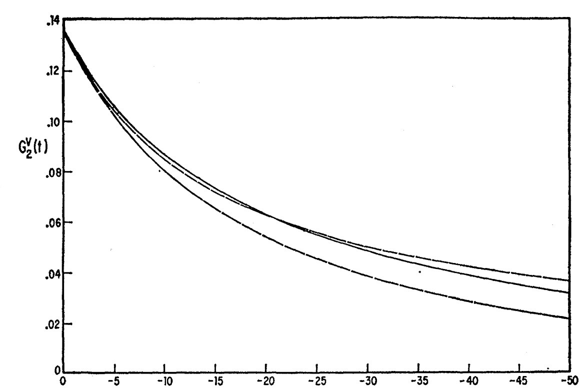

FIG. 8. The anomalous magnetic moment form factor. The solid curve is our result for solution I. The dashed curves are two of the solutions given by De Vries, Hofstadter, and Herman, (reference 5) the lower one being the solution with $a_2=1.15$, $t_2=18$ while the upper is the solution with $a_2=0.96$, $t_2=15$.

IV. THE CALCULATION OF THE FORM FACTORS

The calculation of Γ_i^B, Γ_i, and form factors described in Sec. II were performed with the aid of the CDC 1604 computer of the University of California at San Diego. For each pion-pion solution used the values of the parameters C_N and C_{33} were adjusted to fit $G_1^\rho(0)=\frac{1}{2}a_1e$ and $G_2^\rho(0)=1.83a_2(e/2m)$ as given in Eq. (1.1). The resulting values of C_N and C_{33} for different π-π solutions are given in Table I. It was found that in each case it was possible to find reasonable values of C_N and C_{33} (small and positive) which produced the required values for $G_1^\rho(0)$ and $G_2^\rho(0)$. Furthermore, the form factors produced by each of the π-π solutions had a slope corresponding to t_1 and t_2 in the neighborhood of 20, a substantial shift from the resonance position of 29 used in the calculation. In Fig. 7 and Fig. 8, the calculated form factors are plotted together with the analytic fits to the form factors given in Eq. (1.1). To make this comparison, we use the one-subtraction formulas for the total isovector form factors:

$$G_1^v(t)=[\tfrac{1}{2}e-G_1^\rho(0)+G_1^\rho(t)],$$
$$G_2^v(t)=[1.83(e/2m)-G_2^\rho(0)+G_2^\rho(t)]. \qquad (4.1)$$

V. CONCLUSION

In the present work, there are two factors which contribute to the shifting of the effective ρ mass in the form factor calculation. First, the pion form factor has a peak at ~ 25 while the π-π cross section is peaked at 29. This is due to the broad width of the ρ resonance. Second, the smooth functions $D\Gamma_i$ give a much heavier weight to the low-t part of the spectral function than that of the high-t spectral function. This gives an additional shift from 25 to ~ 20. We believe that these are general features of the two pion contribution to the form factors. The shifting of the effective ρ mass will not be altered substantially in a more sophisticated treatment of the π-π amplitude and the $N\bar{N}\to\pi\pi$ amplitudes.

At the level of our present calculation, we have employed two phenomenological parameters in the treatment of the $N\bar{N}\to\pi\pi$ amplitudes and determined these parameters to fit the form factors. We believe that the $N\bar{N}\to\pi\pi$ amplitudes so obtained can be used in connection with other problems provided they are used only in the $t>4$ region; for example, in the calculation of left-hand discontinuities for πN and NN scattering.[8]

[8] A. Scotti and D. Y. Wong (to be published).